Assessment of Modular Periodicity and Structural Patterns in Various Generalized Fibonacci-Type Sequences

Panchal Nikita V
Research scholar
Department of Mathematics,
VNSGU, Surat, Gujarat, India
Email:- nikita.panchal22@gmail.com

Patel Ripal P
Assistant professor
The Mandvi education society BBA,
BCA and Science college, Mandvi, Gujarat, India
Email:- patelripal82@gmail.com

Shah Devbhadra V
professor
Department of Mathematics,
VNSGU, Surat, Gujarat, India
Email:- dvshah@vnsgu.ac.in

Abstract:

The present study reveals the recent developments in the study of generalized Fibonacci-type sequences and their modular periodicity. Researchers have extended classical results to a wide variety of recurrence relations, including Gaussian Fibonacci numbers, k-Fibonacci and k-Lucas sequences, Jacobsthal-type sequences, and bi-periodic or nonlinear transforms such as squared Pell numbers. A significant focus has been placed on modular reductions of these sequences and the resulting Pisano periods, residue block structures, and distribution properties. Several contributions introduce novel concepts such as residue blocks and bi-periodic recurrence forms, offering refined frameworks for analyzing periodicity modulo integers and powers. The synthesis of algebraic formulations, combinatorial properties, and computational support illustrates the growing depth and applicability of modular periodicity in number theory, cryptography, and algorithmic mathematics. This article presents a cohesive account of such findings and outlines prospective research directions in this vibrant area.

AMS Subject classification: 11B39, 11B50, 11B37

1. Introduction

The Fibonacci sequence $F_n = F_{n-1} + F_{n-2}$ with $F_0 = 0, F_1 = 1; n \geq 2$ and its generalizations continue to play a foundational role in number theory, combinatorics, and discrete mathematics. Beyond their classical formulation, extensions such as k-Fibonacci, Lucas, Jacobsthal, and Pell sequences have received considerable attention, especially in the context of modular arithmetic. The behavior of these sequences under modular reduction – characterized by the emergence of Pisano periods and residue patterns – has prompted a growing body of research, extending the classical results of Wall [15] and Koshy [8].

Modular periodicity, defined as the repeating cycle of a sequence modulo a given integer, offers deep insights into the structure of recurrence sequences. This property has implications not only for theoretical mathematics but also for practical domains such as cryptography, random number generation, and digital signal processing.

Recent studies have expanded the framework by introducing new perspectives such as block structures within periods, quaternionic and Gaussian extensions, and nonlinear transforms such as the squared Pell sequence. These investigations involve both theoretical advancements and algorithmic interpretations, enriching the landscape of modular recurrence analysis. This review collates key contributions to the field, with an emphasis on structural

properties and the evolution of techniques used to analyse periodicity under modular constraints.

2. Review of Literature

The study of the Fibonacci sequence modulo an integer j, often called the Pisano period, reveals that the sequence is purely periodic due to the recurrence relation being fully determined by two consecutive terms. Charles W. and Campbell II [2] investigated the sequence modulo p, a prime and p^k, a prime power, by first investigating the Fibonacci sequence modulo j, yielding a positive integer.

The period, defined as the smallest integer m such that $F_m \equiv 0(mod\ j)$ and $F_{m+1} \equiv 1(mod\ j)$, depends on the prime power decomposition of j, with results showing $m = lcm(m_1, m_2, m_3, \dots, m_s)$, where each m_i is the period modulo a prime power divisor of j.

Using tools from number theory and algebra, including Fermat's Little Theorem, Euler's theorem, quadratic reciprocity, and field theory, the periodicity is investigated for prime and prime power moduli, establishing that if $p \equiv \pm 1(mod\ 5)$, the period divides $p - 1$, while if $p \equiv \pm 2(mod\ 5)$, the period divides $2p + 2$. For higher prime powers, the period modulo p^k divides mp^{k-1}, where m is the period modulo p. These results extend naturally to related linear recursions such as the Lucas sequence, which also exhibits similar periodicity modulo integers.

Matsui, Naito, and Totani [9] investigate the periodicity and residue distribution of the Fibonacci sequence and a related Fibonacci-like sequence under modular reduction. Denoting the reduced Fibonacci sequence modulo m as $F\ (mod\ m)$, the authors analyze the Pisano period and study the frequency distribution of residues within one full period. Pisano period $\pi_k(m)$ represents the length of the repeating cycle of the k-Fibonacci sequence modulo m.

Falcón and Plaza [6] examined the periodicity properties of k-Fibonacci sequences modulo m. The k-Fibonacci sequence $\{F_{k,n}\}_{n \in N}$ is defined as

$$F_{k,n+1} = kF_{k,n} + F_{k,n-1} \text{ for } n \geq 1 \text{ with } F_{k,0} = 0, F_{k,1} = 1,$$

The authors establish that for every odd integer k, the period satisfies the relation $\pi_k(k^2 + 4) = 4(k^2 + 4)$, providing a closed-form expression for the period in this specific case. This result contributes to the broader understanding of modular properties of generalized Fibonacci-type sequences, which play a vital role in areas such as number theory, combinatorics, and cryptographic algorithms.

These topics are classical but continue to yield fascinating insights into the arithmetic behavior of linear recurrence sequences modulo an integer.

Matsui, Naito, and Totani [9] define the Fibonacci-like sequence $B_p(0) = 0, B_p(1) = B_p(2) = 1$ and for natural number n such that $n \geq 2$

$$B_p = B_p(n-1) + \begin{cases} B_p(n-2) + 1, & if\ n = 1. \\ B_p(n-2), & if\ n \neq 1.\end{cases}'$$

for which similar properties – periodicity and residue frequency – are explored. Although proving certain distribution results is analytically challenging, the authors support their work with computational data and conjectures. This dual approach – rigorous mathematics supplemented by computer-aided experimentation – offers a robust foundation for understanding structural behavior in modular recursions beyond the standard Fibonacci case.

Yazlik et al. [16] explored the modular properties of Jacobsthal numbers $\{J_n\}_{n=0}^{\infty}$ and Jacobsthal-Lucas numbers $\{j_n\}_{n=0}^{\infty}$ by examining the sets of remainders modulo m for various positive integers m. These numbers are respectively defined by

$$J_{n+2} = J_{n+1} + 2J_n \quad (J_0 = 0, J_1 = 1)$$
$$j_{n+2} = j_{n+1} + 2j_n \quad (J_0 = 2, j_1 = 1)$$

The study introduces new observations on the periodicity of these sequences modulo m, and importantly, proposes a new method for determining the length of the period. These insights enrich the modular arithmetic perspective of classical recurrence sequences and open potential for algorithmic applications.

Patel and Shah [11] explored the periodic nature of a class of generalized Lucas sequences $\{G_n\}$, defined by the linear recurrence $G_n = aL_{n-2} + bL_{n-1}$, where $\{L_n\}$ denotes the classical Lucas sequence with $L_0 = 2$ and $L_1 = 1$. Their study, focused on the behavior of $\{G_n\}$ modulo powers of $2^e; e \geq 1$, rigorously established that the sequence is periodic for all integers $e \geq 1$. Through algebraic techniques and recurrence analysis, the authors derived several structural results and provided explicit bounds and formulas for the length of the period modulo 2^e. This contribution extends prior research on the modular periodicity of linear recurrence sequences, including the Fibonacci and k-Fibonacci families [6, 14], and adds a new dimension to the study of Lucas-type sequences under modular constraints. The results have implications for understanding Pisano-type periodicities in generalized sequences and their potential arithmetic applications.

Taşcı and Kızılırmak [14] conduct an in-depth analysis of Pisano periods associated with the k-Fibonacci $\{F_{k,n}\}_{n\in\mathbb{N}}$ and k-Lucas sequences $\{L_{k,n}\}_{n\in\mathbb{N}}$ modulo 2^n. The terms of these sequences can be respectively obtained by the Fibonacci-type recurrence relations

$$F_{k,n+1} = k\, F_{k,n} + F_{k,n-1}\ with\ F_{k,0} = 0, F_{k,1} = 1$$

$$L_{k,n+1} = k\, L_{k,n} + L_{k,n-1}\; with\; L_{k,0} = 2, L_{k,1} = 1$$

The authors focus on determining the periodicity of these sequences when reduced modulo powers of 2. They also explore the divisibility characteristics of k-Lucas numbers, providing insights into how their periodic behavior aligns with the structure of the modulus 2^n. The study establishes several results on the minimal period lengths (Pisano periods) for specific values of k and n, thus contributing to the understanding of modular cyclic properties of generalized Fibonacci-type sequences. This work extends classical results like those of Wall and Koshy on $F_n \pmod{m}$, and provides a foundation for further exploration of periodicity and arithmetic functions associated with higher-order recurrence sequences under modular constraints.

Halici and Cerda-Morales [7] examined the algebraic and structural properties of Gaussian Fibonacci numbers, initiating their study with foundational identities before extending the investigation to quaternionic structures whose components are Gaussian Fibonacci numbers. These numbers are respectively defined by the recurrence relations

$$GF_n = GF_{n-1} + GF_{n-2}\; with\; GF_0 = i, GF_1 = 1\; and\; n \geq 2.$$

By employing a Binet-type formula, the authors established several fundamental identities within this hybrid numerical framework, including extensions of classical results such as Cassini's identity to the quaternion-Gaussian context. The work highlights the potential of these quaternions to contribute to broader algebraic and combinatorial structures and suggests future directions involving generalized Gaussian number sequences, particularly the Horadam Gaussian numbers which are recursively defined by the recurrence relation

$$GFQ_n = (GF_n, GF_{n+1}, GF_{n+2}, GF_{n+3}) \in \mathbb{C}^4,$$

with $GFQ_0 = (i, 1, 1 + i, 2 + i)$ and $GFQ_1 = (1, 1 + i, 2 + i, 3 + i)$

This study reinforces the growing relevance of complex-valued and quaternionic extensions of Fibonacci-type sequences in modern algebra and number theory.

Patel and Shah [10] examined the modular periodicity of the Lucas sequence $\{L_n\}$, defined by the recurrence relation $L_n = L_{n-1} + L_{n-2}$; for all $n \geq 2$; with initial terms $L_0 = 2$ and $L_1 = 1$, focusing on the structure of periodic behavior modulo integers $m > 1$. A novel contribution of the study is the introduction of "blocks" – distinct residue groupings within one full period of the Lucas sequence modulo any positive integer $m > 1$. Through rigorous observation and analysis, the authors demonstrated that the period of the Lucas sequence modulo any m is composed of either 1, 2, or 4 such blocks, offering a new perspective on the internal structure of periodic Lucas sequences. This classification enriches the understanding of sequence modularity and may have implications for further exploration of combinatorial and

cryptographic applications. In conclusion, the authors showed that the length of any one period of the Lucas sequence contains either 1, 2, or 4 blocks, introducing a useful framework for analyzing periodic structures in second-order recurrence sequences.

Belbachir and Salhi [1] explore the generalized bi-periodic Fibonacci sequence modulo m, extending the traditional theory of periodicity in linear recursions. The sequence $\{F_n\}_{n\geq 0}$ is defined using a piecewise linear recurrence relation as follows:

$$F_n = \begin{cases} aF_{n-1} + cF_{n-2}, \text{for } n \text{ even} \\ bF_{n-1} + dF_{n-2}, \text{for } n \text{ odd,} \end{cases}$$

with initial conditions $F_0 = 0, F_1 = 1$, where $a, b, c, d \in \mathbb{Z}^+$.

The main focus of the paper is the investigation of the periodicity modulo m of this bi-periodic sequence, under the condition that $m \geq 2$ and $\gcd(m, cd) = 1$. The authors extend classical results on the Pisano period and the rank of apparition (least index n such that $m \mid F_n$) to the context of this more general recurrence. Their findings reveal structural periodic properties that distinguish bi-periodic sequences from their classical Fibonacci counterparts, offering new insights into congruence behavior and modular arithmetic of such generalized sequences.

Josep and Joan-C [13] explore the Pisano periods of generalized Fibonacci sequences, focusing on the behavior of r-generalized Fibonacci numbers modulo m. Extending the classical notion where Fibonacci numbers reduced modulo m form purely periodic sequences, the study considers higher-order recurrences defined by $F_n = F_{n-1} + F_{n-2} + F_{n-3} + \cdots .. + F_{n-r}$ with suitable initial conditions. The authors analyse periodicity properties, determine upper bounds on Pisano periods in terms of m and r, and establish connections with linear recurrences over finite rings and fields. In particular, the paper provides results on minimal period lengths, divisibility conditions, and structural characteristics of these generalized sequences under modular reduction, highlighting both similarities and differences with the classical Fibonacci case. These findings enrich the algebraic theory of recurrence sequences, with implications for number theory, combinatorics, and cryptographic applications.

Ömür Deveci, Yeşim Aküzüm, and Gamze Özdemir [5] investigated the complex-type Fibonacci p-sequences within the framework of finite groups and modular arithmetic. The authors first define the sequence using complex numbers as under:

$$F_{p,i}^*(n + p + 1) = i^{p+1} \cdot F_{p,i}^*(n + p) + i \cdot F_{p,i}^*(n); n \geq 0$$
$$\text{where, } F_{p,i}^*(0) = \cdots . = F_{p,i}^*(p - 1) = 0, F_{p,i}^*(p) = 1 \text{ and } \sqrt{-1} = i.$$

Then they study its behavior modulo m, showing that such sequences are always periodic and derived their periods, ranks, and orders. They then analyzed the generating matrix of these sequences under modulo reduction, demonstrating connections with cyclic groups and established relationships between the sequence's periods and the orders of the resulting groups. Extending the study to group theory, the paper redefines the complex-type Fibonacci p-sequence in terms of group elements and proves that its orbits in finite groups are periodic. As a concrete application, the authors compute and discuss the periods of the complex-type Fibonacci 2-sequence in the dihedral group D_{2m}, highlighting the structural links between generalized Fibonacci sequences, modular periodicity, and group theory. This work provides a unified perspective combining number theory, recurrence relations, and algebraic group structures, thereby enriching the study of generalized Fibonacci-type sequences.

Patel and Shah [12] conducted a detailed investigation into the modular properties of the squared Pell sequence $SP_n = P_n^2$, where $\{P_n\}$ denotes the classical Pell numbers defined by the recurrence $P_n = 2P_{n-1} + P_{n-2}$ with initial terms $P_0 = 0, \ P_1 = 1$. Specifically, the authors examined the periodic behavior of $\{SP_n\}$ modulo powers of ten, namely 10^e, and introduced the novel concept of "block structures" to describe repeating residue patterns within a full period. Their findings reveal that each period modulo 10^e comprises either one or two distinct residue blocks, a structural regularity not previously identified in similar modular studies. This contributes to the broader understanding of periodicity in nonlinear transforms of classical sequences, complementing earlier work on the modular behavior of k-Fibonacci [6, 14], Jacobsthal and Jacobsthal–Lucas sequences [4, 16], and generalized recurrence sequences over residue classes [1]. Such analysis enriches the theoretical landscape of recurrence sequences under modular constraints and holds promise for applications in number theory, coding theory, and computational mathematics.

Costa, E.A., Spreafico, E.V.P., and Catarino, P.M.M. [3] introduced the sequence of generalized bi-periodic Lucas-balancing numbers, which extends the classical Lucas-balancing sequence by incorporating bi-periodic recurrence relations. Motivated by previous generalizations of integer sequences such as the bi-periodic Fibonacci, Lucas, Pell–Lucas, Jacobsthal, and Leonardo sequences, the authors defined this new class with arbitrary initial conditions and examined its structural properties. They established recurrence relations, generating functions, and Binet-type formulas under two cases depending on the discriminant of the associated characteristic polynomial. Furthermore, several well-known identities, including those of Tagiuri–Vajda, d'Ocagne, Catalan, and Cassini, were generalized to this

new framework. Their work significantly enriches the algebraic and analytic theory of Lucas-balancing numbers and highlights how the combination of balancing numbers with bi-periodic structures leads to deeper generalizations and potential applications in number theory and combinatorial analysis.

3. Concluding Remarks

The literature surveyed herein underscores the richness and diversity of modular periodicity in generalized Fibonacci-type sequences. From the exploration of Gaussian and quaternionic extensions to the structural characterization of residue blocks in Lucas and Pell sequences, the field has seen a broadening of both theoretical frameworks and analytical methods. Notably, the identification of block structures, the closed-form expressions for specific Pisano periods, and modular behavior of bi-periodic and nonlinear sequences reflect a trend toward unifying structural insights with modular arithmetic. The convergence of algebraic methods, computational verification, and combinatorial techniques has enabled deeper understanding of periodicity and residue distribution across various recurrence families.

Future research directions may include further exploration of higher-order recursions, rank of apparition for complex sequences, and applications in cryptographic key generation and pseudorandom structures. As the field continues to mature, the interplay between generalized recurrence relations and modular constraints remains a fertile ground for mathematical inquiry.

References:

1. Belbachir, H., & Salhi, C. (2021). The generalized bi-periodic Fibonacci sequence modulo *m*. *Journal of Integer Sequences, 24*, Article 21.3.1.

2. Campbell, C. W. II. (2007). *The period of the Fibonacci sequence modulo j* (Undergraduate research project, Math 399). Advisor: Dr. Nick Rogers.

3. Costa, E. A., Spreafico, E. V. P., & Catarino, P. M. M. (2025). A note on the generalized bi-periodic Lucas-balancing numbers. *Brazilian Electronic Journal of Mathematics, 6*, e75273.

4. Deveci, O., Karaduman, E., & Sağlam, G. (2016). The Jacobsthal sequences in finite groups. *Bulletin of the Iranian Mathematical Society, 42*(1), 79–89.

5. Deveci, Ö., Aküzüm, Y., & Özdemir, G. (2024). The complex-type Fibonacci p-sequences in finite groups. *Journal of Science and Arts, 24*(3), 505–514.

6. Falcón, S., & Plaza, A. (2009). k-Fibonacci sequences modulo *m*. *Chaos, Solitons & Fractals, 41*, 497–504.

7. Halici, S., & Cerda-Morales, G. (2021). On quaternion-Gaussian Fibonacci numbers and their properties. *Analele Ştiinţifice ale Universităţii Ovidius Constanţa, 29*(1), 71–82.

8. Koshy, T. (2001). *Fibonacci and Lucas numbers with applications*. John Wiley & Sons.

9. Matsui, H., Naito, M., & Totani, N. (2010). The period and the distribution of the Fibonacci-like sequence under various moduli. *Rose-Hulman Undergraduate Mathematics Journal, 10*(1).

10. Patel, R. P., & Shah, D. V. (2017). Periodicity of generalized Lucas numbers and the length of its period under modulo 2^e. *Mathematics Today, 33*, 67–74.

11. Patel, R. P., & Shah, D. V. (2021). Blocks within the period of Lucas sequence. *Ratio Mathematica, 41*, 71–78.

12. Patel, R., & Shah, D. (2025). Periodic patterns and block structures in squared Pell sequence modulo 10^e. *International Journal of Research and Scientific Innovation, 12*(5).

13. Sroysang, B. (2022). On the period of Fibonacci sequence modulo *n*. *arXiv*. https://arxiv.org/abs/2204.00234

14. Taşcı, D., & Özkan Kızılırmak, G. (2018). Pisano periods for the k-Fibonacci and k-Lucas sequences mod 2n. *Gazi University Journal of Science, 31*(1), 213–220.

15. Wall, D. D. (1960). Fibonacci series modulo *m*. *American Mathematical Monthly, 67*(6), 525–532.

16. Yazlik, Y., Yilmaz, N., Taşkara, N., & Uslu, K. (2016). Jacobsthal family modulo *m*. *TWMS Journal of Applied and Engineering Mathematics, 6*(1), 15–21.